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Multiplicative partition functions for reverse plane partitions derived from an integrable dynamical system

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Abstract. In this paper we clarify a close connection between reverse plane partitions and an integrable dynamical system called the discrete two-dimensional (2D) Toda molecule. We show that a multiplicative partition function for reverse plane partitions of arbitrary shape with bounded parts can be obtained from each non-vanishing solution to the discrete 2D Toda molecule. As an example we derive a partition function which generalizes MacMahon's triple product formula and Gansner's multi-trace generating function from a specific solution to the discrete 2D Toda molecule.

Keywords: reverse plane partitions, lattice paths, integrable systems

1 Introduction

Let λ be an (integer) partition or the corresponding Young diagram. A *reverse plane partition* of shape λ is a filling of cells in λ with nonnegative integers such that all rows and columns are weakly increasing. One of the most prominent results in the study of (reverse) plane partitions is the discovery of *multiplicative* generating functions, namely those which can be nicely factored.

The first discovery is due to MacMahon [8] who proved the product formula

$$\sum_{\pi} q^{|\pi|} = \prod_{i=1}^{r} \prod_{j=1}^{c} \prod_{k=1}^{n} \frac{1-q^{i+j+k-1}}{1-q^{i+j+k-2}} \tag{1.1}$$

for plane partitions π of $r \times c$ rectangular shape with parts at most n. MacMahon's study on plane partitions was revived by Stanley. Among his vast amounts of results a multiplicative generating function involving the trace statistic is of great importance [11]. Gansner [1] later refined Stanley's trace generating function into the multi-trace one

$$\sum_{\pi \in \mathrm{RPP}(\lambda)} \prod_{\ell=1-r}^{c-1} x_\ell^{\mathrm{tr}_\ell(\pi)} = \prod_{(i,j)\in\lambda} \left(1 - \prod_{\ell=j-\lambda'_j}^{\lambda_i - i} x_\ell\right)^{-1} \tag{1.2}$$

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where $\mathrm{RPP}(\lambda)$ denotes the set of reverse plane partitions $\pi = (\pi_{i,j})$ of shape λ with r rows and c columns, $\mathrm{tr}_\ell(\pi) = \sum_{-i+j=\ell} \pi_{i,j}$, the ℓ-trace, and λ' the shape conjugate with λ. Okada [10] further generalizes (1.2) based on Macdonald symmetric polynomials and Schur processes. Note that plane partitions considered in (1.1) are those with bounded parts but reverse plane partitions in (1.2) are those with unbounded parts.

In this paper we clarify a close connection between reverse plane partitions and an integrable dynamical system, called the *discrete two-dimensional (2D) Toda molecule* [3]. We show that a multiplicative partition function for reverse plane partitions can be derived from each non-vanishing solution to the dynamical system (Theorem 6 in Section 4) where reverse plane partitions considered are those of arbitrary shape with bounded parts. As a concrete example we derive a partition function which generalizes both MacMahon's product formula (1.1) and Gansner's multi-trace generating function (1.2) from a specific solution (Theorem 7 in Section 5). The key idea comes from a combinatorial interpretation of the discrete 2D Toda molecule in terms of non-intersecting lattice paths (Section 3). Note that Viennot [12] takes a similar approach to count non-intersecting Dyck paths by using the quotient-difference (qd) algorithm for Padé approximation.

We remark that the author showed in his previous papers [4, 5] a similar result for plane partitions of rectangular shape by means of biorthogonal polynomials. The discrete 2D Toda molecule is equivalent to the adjacent relations among biorthogonal polynomials used in [4, 5], see also, e.g., [9]. The technique developed in this paper has the same ability as the previous one by biorthogonal polynomials, though how to deal with the case of non-rectangular shape is not discussed in [4, 5].

2 Solutions to the discrete 2D Toda molecule

We show a brief review on the integrable dynamical system discussed throughout the paper. The discrete two-dimensional (2D) Toda molecule is one of the most typical integrable dynamical systems that was introduced as a discrete analogue of the Toda lattice [3]. The evolution of the discrete 2D Toda molecule is described by the difference equations

$$a_n^{(s,t+1)} + b_n^{(s+1,t)} = a_n^{(s,t)} + b_{n+1}^{(s,t)}, \tag{2.1a}$$

$$a_n^{(s,t+1)} b_{n+1}^{(s+1,t)} = a_{n+1}^{(s,t)} b_{n+1}^{(s,t)}, \tag{2.1b}$$

$$(s,t) \in \mathbb{Z}^2, \quad n \in \mathbb{Z}_{\geq 0}, \quad b_0^{(s,t)} = 0. \tag{2.1c}$$

Through a dependent variable transformation

$$a_n^{(s,t)} = \frac{\tau_{n+1}^{(s+1,t)}\tau_n^{(s,t)}}{\tau_n^{(s+1,t)}\tau_{n+1}^{(s,t)}}, \qquad b_n^{(s,t)} = \frac{\tau_{n-1}^{(s,t+1)}\tau_{n+1}^{(s,t)}}{\tau_n^{(s,t+1)}\tau_n^{(s,t)}} \tag{2.2}$$

with $\tau_0^{(s,t)} = 1$ we obtain from (2.1)

$$\tau_{n-1}^{(s+1,t+1)}\tau_{n+1}^{(s,t)} - \tau_n^{(s+1,t+1)}\tau_n^{(s,t)} + \tau_n^{(s+1,t)}\tau_n^{(s,t+1)} = 0, \tag{2.3a}$$

$$(s,t) \in \mathbb{Z}^2, \quad n \in \mathbb{Z}_{\geq 1}, \quad \tau_0^{(s,t)} = 1, \tag{2.3b}$$

that is the so-called bilinear form of the discrete 2D Toda molecule.

The determinant

$$\tau_n^{(s,t)} = \det_{0\leq i,j<n}(f_{s+i,t+j}) = \begin{vmatrix} f_{s,t} & \cdots & f_{s,t+j} & \cdots & f_{s,t+n-1} \\ \vdots & & \vdots & & \vdots \\ f_{s+i,t} & \cdots & f_{s+i,t+j} & \cdots & f_{s+i,t+n-1} \\ \vdots & & \vdots & & \vdots \\ f_{s+n-1,t} & \cdots & f_{s+n-1,t+j} & \cdots & f_{s+n-1,t+n-1} \end{vmatrix} \tag{2.4}$$

solves the bilinear form (2.3) where $f = f_{i,j}$ is an arbitrary function defined on $\mathbb{Z}^2$. Therefore $a_n^{(s,t)}$, $b_n^{(s,t)}$ given by (2.2) with (2.4) solves the discrete 2D Toda molecule (2.1) if the determinant does not vanish. Conversely there exists a function f on $\mathbb{Z}^2$ which satisfies (2.2) with (2.4) for every solution $a_n^{(s,t)} \neq 0$, $b_n^{(s,t)} \neq 0$ to (2.1). We have the following correspondence between $a_n^{(s,t)}$, $b_n^{(s,t)}$ and f.

Proposition 1. *For each solution $a_n^{(s,t)} \neq 0$, $b_n^{(s,t)} \neq 0$ to the discrete 2D Toda molecule (2.1) there exists a function $f = f_{i,j}$ on $\mathbb{Z}^2$ which gives the same solution through (2.2) with (2.4). Moreover such an f is uniquely determined up to the transformation $f_{i,j} \to \varphi_j f_{i,j}$ by any non-vanishing function $\varphi = \varphi_j$ on $\mathbb{Z}$.*

Giving a non-vanishing solution $a_n^{(s,t)}$, $b_n^{(s,t)}$ to the discrete 2D Toda molecule is thus essentially equivalent to giving a function f on $\mathbb{Z}^2$.

3 Lattice path combinatorics

In this paper we use a matrix-like coordinate to draw a square lattice $\mathbb{Z}^2$ where the nearest neighbors $(i+1,j)$, $(i-1,j)$, $(i,j+1)$ and $(i,j-1)$ of a lattice point (i,j) are located on the south, north, east and west of (i,j) respectively. We call a subset $\mathbb{L}$ of $\mathbb{Z}^2$ *regular* such that (i) if $(i,j) \in \mathbb{L}$ then $(i+k, j+k) \in \mathbb{L}$ for all $k \geq 1$; (ii) if $(i,j) \in \mathbb{L}$ then

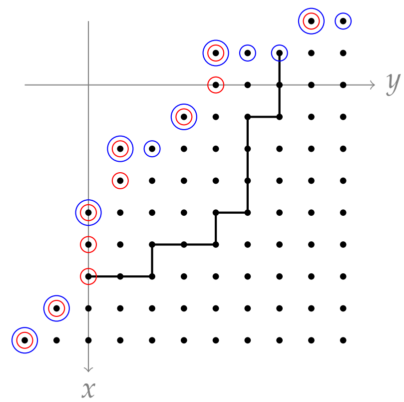

Figure 1: A regular subset $\mathbb{L}$ of $\mathbb{Z}^2$ and a lattice path. North and west boundary points are marked blue and red respectively. Convex corners are those marked in both the colors.

$(i-k,j) \notin \mathbb{L}$ and $(i,j-k) \notin \mathbb{L}$ for some $k \geq 1$. We call a point $(i,j) \in \mathbb{L}$ a *north boundary point* if $(i-1,j) \notin \mathbb{L}$; similarly a *west boundary point* if $(i,j-1) \notin \mathbb{L}$. We call a point $(i,j) \in \mathbb{L}$ a *convex corner* if (i,j) is a north and west boundary point. The interest is in lattice paths on a regular subset $\mathbb{L}$ of $\mathbb{Z}^2$ consisting of north and east steps. See Figure 1 for an example.

We think of a regular subset $\mathbb{L}$ of $\mathbb{Z}^2$ as a graph with vertices $\mathbb{L}$ and edges connecting nearest neighbors. We determine the weights of edges by using a solution $a_n^{(s,t)}$, $b_n^{(s,t)}$ to the discrete 2D Toda molecule (2.1) as follows.

(a) The vertical edge with north endpoint at (i,j) has the weight $a_n^{(i-n,j-n)}$ if $(i-n,j-n)$ is a west boundary point of $\mathbb{L}$.

(b) The vertical edge with south endpoint at (i,j) has the weight $b_n^{(i-n,j-n)}$ if $(i-n,j-n)$ is a north boundary point of $\mathbb{L}$.

(c) Every horizontal edge has the unit weight 1.

See Figure 2 for an example. We define the weight $w(\mathbb{L};a,b;P)$ of a lattice path P on $\mathbb{L}$ to be the product of the weights of all the edges passed by P. We conventionally consider empty paths P with no steps for which $w(\mathbb{L};a,b;P)=1$. For $(i,j) \in \mathbb{L}$ and $(k,\ell) \in \mathbb{L}$ we define

$$g(\mathbb{L};a,b;i,j;k,\ell) = \sum_P w(\mathbb{L};a,b;P) \tag{3.1}$$

where the sum ranges over all the lattice paths on $\mathbb{L}$ going from (i,j) to (k,ℓ).

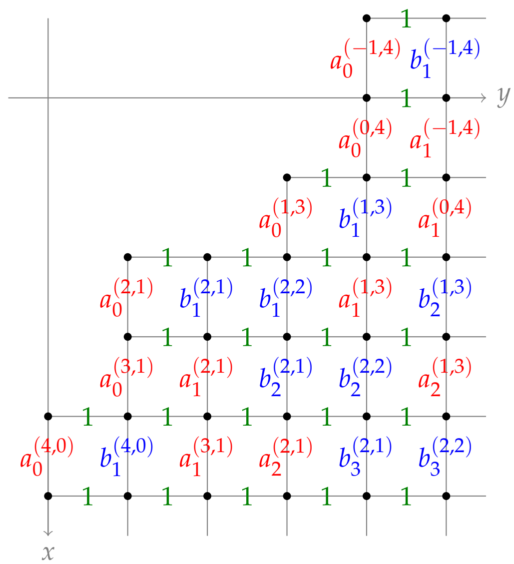

Figure 2: The weights of edges.

Let $\mathsf{x}(j)$ denote the x-coordinate (or the vertical —) of the north boundary point of $\mathbb{L}$ with y-coordinate (or horizontal —) equal to j; let $\mathsf{y}(i)$ the y-coordinate of the west boundary point of $\mathbb{L}$ with x-coordinate equal to i. The following theorem gives a combinatorial interpretation of the discrete 2D Toda molecule and refines Proposition 1.

Theorem 2. *Let $a_n^{(s,t)} \neq 0$, $b_n^{(s,t)} \neq 0$ be a solution to the discrete 2D Toda molecule (2.1). Assume that a function $f = f_{i,j}$ on $\mathbb{Z}^2$ gives the same solution through (2.2) with (2.4). Let $\mathbb{L}$ be a regular subset of $\mathbb{Z}^2$. Then for any $(i,j) \in \mathbb{L}$,*

$$\frac{f_{i,j}}{f_{\mathsf{x}(j),j}} = g(\mathbb{L}; a, b; i, \mathsf{y}(i); \mathsf{x}(j), j). \tag{3.2}$$

In order to prove the theorem we use the following lemma.

Lemma 3. *Let $\mathbb{L}$ be a regular subset of $\mathbb{Z}^2$ with a convex corner $(s,t) \in \mathbb{L}$. Let $\mathbb{L}'$ denote the regular subset of $\mathbb{Z}^2$ obtained from $\mathbb{L}$ by deleting the convex corner (s,t). Then for any (i,j) and (k,ℓ) in $\mathbb{L}'$ with $i-j \neq s-t$ and $k-\ell \neq s-t$,*

$$g(\mathbb{L}; a, b; i, j; k, \ell) = g(\mathbb{L}'; a, b; i, j; k, \ell). \tag{3.3}$$

Proof. The difference between $\mathbb{L}$ and $\mathbb{L}'$ is only in the existence and the absence of the convex corner (s,t), and the weights of vertical edges between the two diagonal lines $d_- : y - x = t - s - 1$ and $d_+ : y - x = t - s + 1$. (The vertical edges between d_- and d_+ are weighted by $a_n^{(s,t)}$, $b_n^{(s,t)}$ on $\mathbb{L}$ and by $a_n^{(s,t+1)}$, $b_n^{(s+1,t)}$ on $\mathbb{L}'$.) Assume that $i - j \neq s - t$

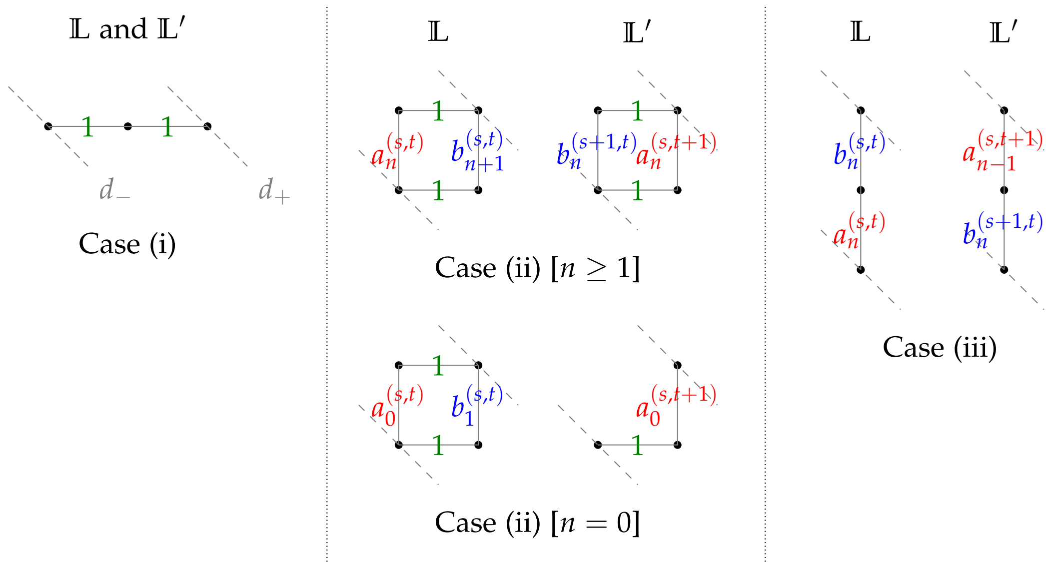

Figure 3: Proof of Lemma 3.

and $k - \ell \neq s - t$ meaning that (i, j) and (k, ℓ) is outside the region between d_- and d_+. (Those may be on $d_\pm$.) If both (i, j) and (k, ℓ) are either in the south of d_- or in the north of d_+ then the identity (3.3) clearly holds since lattice paths going from (i, j) to (k, ℓ) never enter the region between d_- and d_+. In the rest of the proof we thus assume that (i, j) is in the south of d_- and (k, ℓ) in the north of d_+.

Each lattice path P going from (i, j) to (k, ℓ) is uniquely divided into three subpaths: P_- from (i, j) to d_-, Q of two steps between d_- and d_+ and P_+ from d_+ to (k, ℓ). Obviously $w(P_\pm) = w'(P_\pm)$ where w and w' are abbreviations of $w(\mathbb{L}; a, b; \cdot)$ and $w(\mathbb{L}'; a, b; \cdot)$ respectively. The proof of (3.3) thus amounts to showing that $g(i, j; k, \ell) = g'(i, j; k, \ell)$ for each (i, j) on d_- and (k, ℓ) on d_+ where g and g' are abbreviations of $g(\mathbb{L}; a, b; \cdot)$ and $g(\mathbb{L}'; a, b; \cdot)$ respectively. Since Q is of two steps we have only three cases: (i) $(i, j) = (s + n, t + n - 1)$ and $(k, \ell) = (s + n, t + n + 1)$ for some $n \geq 1$; (ii) $(i, j) = (s + n + 1, t + n)$ and $(k, \ell) = (s + n, t + n + 1)$ for some $n \geq 0$; (iii) $(i, j) = (s + n + 1, t + n)$ and $(k, \ell) = (s + n - 1, t + n)$ for some $n \geq 1$. See Figure 3.

Case (i): The unique lattice path going from $(i, j) = (s + n, t + n - 1)$ to $(k, \ell) = (s + n, t + n + 1)$ of two east steps is both on $\mathbb{L}$ and on $\mathbb{L}'$. Thus $g(i, j; k, \ell) = g'(i, j; k, \ell) = 1$.

Case (ii): There are two lattice paths going from $(i, j) = (s + n + 1, t + n)$ to $(k, \ell) = (s + n, t + n + 1)$ one of which is Q_1 going north then east, the other is Q_2 going east then north. If $n \geq 1$ then Q_1 and Q_2 are both on $\mathbb{L}$ and on $\mathbb{L}'$, and $w(Q_1) = a_n^{(s,t)}$, $w(Q_2) = b_{n+1}^{(s,t)}$, $w'(Q_1) = a_n^{(s,t+1)}$ and $w'(Q_2) = b_n^{(s+1,t)}$. Thus $g(i, j; k, \ell) = a_n^{(s,t)} + b_{n+1}^{(s,t)} = a_n^{(s,t+1)} + b_n^{(s+1,t)} = g'(i, j; k, \ell)$ because of (2.1a). If $n = 0$ then Q_1 and Q_2 are on $\mathbb{L}$ while only Q_2 on $\mathbb{L}'$, and $w(Q_1) = a_0^{(s,t)}$, $w(Q_2) = b_1^{(s,t)}$ and $w'(Q_1) = a_0^{(s,t+1)}$. (Q_1 is not on $\mathbb{L}'$ since Q_1 passes through $(s,t) \notin \mathbb{L}'$.) Thus $g(i,j;k,\ell) = a_0^{(s,t)} + b_1^{(s,t)} = a_0^{(s,t+1)} = g'(i,j;k,\ell)$ because of (2.1a) with (2.1c).

Case (iii): The unique lattice path going from $(i,j) = (s+n+1,t+n)$ to $(k,\ell) = (s+n-1,t+n)$ of two north steps is both on $\mathbb{L}$ and on $\mathbb{L}'$. The weight of the lattice path is $a_n^{(s,t)} b_n^{(s,t)}$ on $\mathbb{L}$ and $a_{n-1}^{(s,t+1)} b_n^{(s+1,t)}$ on $\mathbb{L}'$. Thus $g(i,j;k,\ell) = a_n^{(s,t)} b_n^{(s,t)} = a_{n-1}^{(s,t+1)} b_n^{(s+1,t)} = g'(i,j;k,\ell)$ because of (2.1b). □

Proof of Theorem 2. Let $\mathbb{L}'$ denote the regular subset of $\mathbb{Z}^2$ defined by $\mathbb{L}' = \mathbb{L} \setminus \{(s,t) \in \mathbb{L};\ s < i \text{ and } t < j\}$. From Lemma 3 then $g(\mathbb{L};a,b;i,\mathsf{y}(i);\mathsf{x}(j),j) = g(\mathbb{L}';a,b;i,\mathsf{y}(i);\mathsf{x}(j),j)$ because we can obtain $\mathbb{L}'$ from $\mathbb{L}$ by deleting convex corners iteratively. A lattice path going from $(i,\mathsf{y}(i))$ to $(\mathsf{x}(j),j)$ on $\mathbb{L}'$ is unique because such a lattice path cannot turn north until (i,j) and cannot turn east from (i,j). The weight of the unique lattice path on $\mathbb{L}'$ implies that $g(\mathbb{L}';a,b;i,\mathsf{y}(i);\mathsf{x}(j),j) = \prod_{k=\mathsf{x}(j)}^{i-1} a_0^{(k,j)}$. The last product is equal to $f_{i,j}/f_{\mathsf{x}(j),j}$ because $a_0^{(k,j)} = f_{k+1,j}/f_{k,j}$ from (2.2) and (2.4). □

Theorem 2 admits a combinatorial interpretation of the determinant $\tau_n^{(s,t)}$ by means of Gessel–Viennot–Lindström's method [2, 7]. For $(s,t) \in \mathbb{L}$ and $n \geq 0$ we define $\mathrm{LP}(\mathbb{L},s,t,n)$ to be the set of n-tuples $(P_0,\dots,P_{n-1})$ of lattice paths on $\mathbb{L}$ such that (i) P_k goes from $(s+k,\mathsf{y}(s)+k)$ to $(\mathsf{x}(t)+k,t+k)$ for each $0 \leq k < n$, and (ii) $P_0,\dots,P_{n-1}$ are *non-intersecting*: $P_j \cap P_k = \emptyset$ if $j \neq k$. See Figure 4 where the second figure shows such an n-tuple of non-intersecting lattice paths.

Theorem 4. *Let $a_n^{(s,t)} \neq 0$, $b_n^{(s,t)} \neq 0$ be a solution to the discrete 2D Toda molecule (2.1). Assume that a function $f = f_{i,j}$ on $\mathbb{Z}^2$ gives the same solution through (2.2) with (2.4). Let $\mathbb{L}$ be a regular subset of $\mathbb{Z}^2$. Then for any $(s,t) \in \mathbb{L}$ and $n \geq 0$,*

$$\frac{\tau_n^{(s,t)}}{\tau_n^{(\mathsf{x}(t),t)}} = \sum_{(P_0,\dots,P_{n-1}) \in \mathrm{LP}(\mathbb{L},s,t,n)} \prod_{k=0}^{n-1} w(\mathbb{L};a,b;P_k) \tag{3.4}$$

where $\tau_n^{(s,t)} = \det_{0 \leq i,j < n}(f_{s+i,t+j})$.

Proof. Gessel–Viennot–Lindström's method implies from Theorem 2 that

$$\frac{\tau_n^{(s,t)}}{\prod_{k=0}^{n-1} f_{\mathsf{x}(t+k),t+k}} = \sum_{(P'_0,\dots,P'_{n-1})} \prod_{k=0}^{n-1} w(\mathbb{L};a,b;P'_k) \tag{3.5}$$

where the sum ranges over all n-tuples $(P'_0,\dots,P'_{n-1})$ of non-intersecting lattice paths on $\mathbb{L}$ such that P'_k goes from $(s+k,\mathsf{y}(s+k))$ to $(\mathsf{x}(t+k),t+k)$ for each $0 \leq k < n$. Eliminating the steps frozen due to the non-intersecting condition we obtain $(P_0,\dots,P_{n-1}) \in \mathrm{LP}(\mathbb{L},s,t,n)$, see Figure 4 for example. The weight of the eliminated frozen steps is

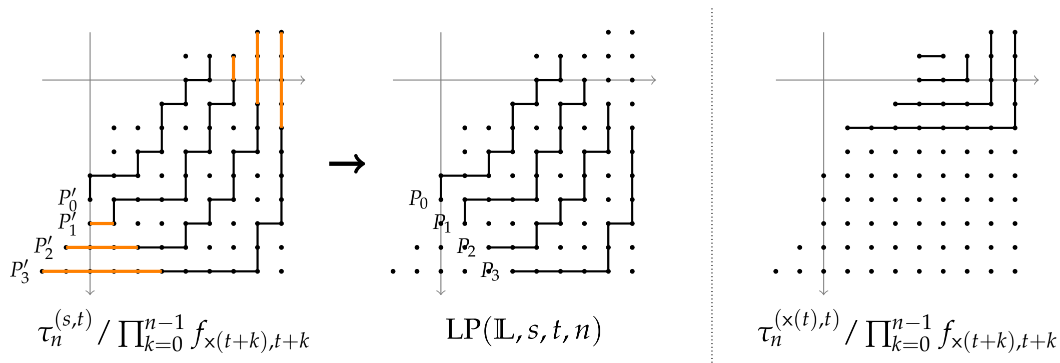

Figure 4: Proof of Theorem 4 where $(s,t) = (5,4)$ and $n = 4$. In the first figure the steps in orange are frozen due to the non-intersecting condition.

equal to the weight of the unique configuration of non-intersecting lattice paths counted in $\tau_n^{(\times(t),t)} / \prod_{k=0}^{n-1} f_{\times(t+k),t+k}$, see the last figure in Figure 4 for example. Thus $\tau_n^{(s,t)}$ is equal to the right-hand side of (3.4) multiplied by $\tau_n^{(\times(t),t)}$. □

Note that the left-hand side of (3.4) can be expressed as

$$\frac{\tau_n^{(s,t)}}{\tau_n^{(\times(t),t)}} = \prod_{i=1}^{s-\times(t)} \prod_{k=1}^{n} a_{k-1}^{(s-i,t)} \tag{3.6}$$

from (2.2). We can readily evaluate the sum in (3.4), a partition function for non-intersecting lattice paths, by using this formula.

4 A multiplicative partition function for reverse plane partitions

Let λ be a partition and let $n \geq 0$. We write $\mathrm{RPP}(\lambda, n)$ for the set of reverse plane partitions of shape λ with parts at most n. Let r and c denote the numbers of rows and columns in λ respectively. We then define a regular subset $\mathbb{L}(\lambda)$ of $\mathbb{Z}^2$ by

$$\mathbb{L}(\lambda) = \{(i,j) \in \mathbb{Z}^2_{\geq 0};\ j \geq c - \lambda_{r-i}\} \tag{4.1}$$

where λ_i denotes the i-th part of λ for $1 \leq i \leq r$ and $\lambda_i = c$ for $i \leq 0$. There is a bijection between $\mathrm{LP}(\mathbb{L}(\lambda), r, c, n)$ and $\mathrm{RPP}(\lambda, n)$ which is described as follows. Given an n-tuple $(P_0, \ldots, P_{n-1}) \in \mathrm{LP}(\mathbb{L}(\lambda), r, c, n)$ of non-intersecting lattice paths on $\mathbb{L}(\lambda)$,

(i) move the lattice path P_k northwest by $(-k, -k)$ for each $0 \leq k < n$;

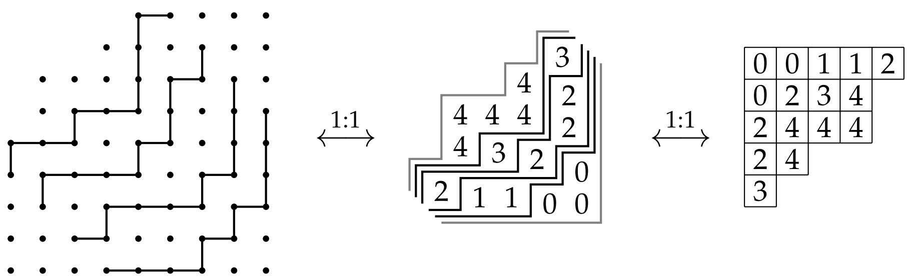

Figure 5: The bijection between $\mathrm{LP}(\mathbb{L}(\lambda), r, c, n)$ and $\mathrm{RPP}(\lambda, n)$ where $\lambda = (5, 4, 4, 2, 1)$ with $r = 5$ rows and $c = 5$ columns, and $n = 4$.

(ii) fill in the cells between P_{n-k-1} and P_{n-k} with k for each $0 \le k \le n$ where P_{-1} is the lattice path going from $(r, 0)$ to $(0, c)$ along the border of $\mathbb{L}(\lambda)$ and P_n is that going east from $(r, 0)$, turning north at (r, c) and going north to $(0, c)$;

(iii) rotate 180° to obtain a reverse plane partition in $\mathrm{RPP}(\lambda, n)$.

Figure 5 demonstrates the bijection by an example. This bijection is essentially the same as the classical interpretation of plane partitions by "zig-zag" non-intersecting paths [6].

We set up weight for reverse plane partitions which is equivalent to the weight for lattice paths defined in Section 3. Let $\lambda' = (\lambda'_1, \dots, \lambda'_c)$ denote the partition conjugate with λ. We define $\alpha_{i,j}$ by

$$\alpha_{i+k,\lambda_i+k} = a_{n-k-1}^{(r-i,c-\lambda_i)}, \qquad \alpha_{\lambda'_j+k,j+k-1} = b_{n-k}^{(r-\lambda'_j,c-j)} \tag{4.2}$$

for $1 \le i \le r$, $1 \le j \le c$ and $k < n$ where $a_n^{(s,t)} \ne 0$, $b_n^{(s,t)} \ne 0$ is a solution to the discrete 2D Toda molecule (2.1). We then define the weight of a reverse plane partition π by

$$v(\lambda, n; a, b; \pi) = \prod_{(i,j)\in\lambda} \prod_{k=1}^{\pi_{i,j}} \frac{\alpha_{i+k-1,j+k-2}}{\alpha_{i+k-1,j+k-1}}. \tag{4.3}$$

Lemma 5. *Let λ be a partition with r rows and c columns, and let $n \ge 0$. Assume that $\pi \in \mathrm{RPP}(\lambda, n)$ and $(P_0, \dots, P_{n-1}) \in \mathrm{LP}(\mathbb{L}(\lambda), r, c, n)$ correspond to each other by the bijection. Then*

$$v(\lambda, n; a, b; \pi) = \frac{\prod_{k=0}^{n-1} w(\mathbb{L}(\lambda); a, b; P_k)}{\prod_{i=1}^{r} \prod_{k=1}^{n} a_{k-1}^{(r-i,c-\lambda_i)}}. \tag{4.4}$$

Sketch of proof. Actually $\alpha_{i,j}$ is defined so that $v(\pi) = v(\lambda, n; a, b; \pi)$ is proportional to $\prod_{k=0}^{n-1} w(P_k)$ with $w(P) = w(\mathbb{L}(\lambda); a, b; P)$. That is, there exists a constant κ such that $v(\pi) = \kappa \prod_{k=0}^{n-1} w(P_k)$. From (4.3), $v(\lambda, n; a, b; \pi^{\varnothing}) = 1$ for the *empty* reverse plane partition $\pi^{\varnothing} \in \mathrm{RPP}(\lambda, n)$ whose parts are all 0. Thus $\kappa^{-1} = \prod_{k=0}^{n-1} w(P_k^{\varnothing})$ where $(P_0^{\varnothing}, \dots, P_{n-1}^{\varnothing}) \in \mathrm{LP}(\mathbb{L}(\lambda), r, c, n)$ corresponds to $\pi^{\varnothing}$ by the bijection. We observe that $P_0^{\varnothing}$ goes from $(r, 0)$ to $(0, c)$ along the border of $\mathbb{L}(\lambda)$, and $P_1^{\varnothing}, \dots, P_{n-1}^{\varnothing}$ are copies of $P_0^{\varnothing}$. Here $w(P_k^{\varnothing}) = \prod_{i=1}^{r} a_k^{(r-i, c-\lambda_i)}$ and hence κ^{-1} is equal to the denominator of the right-hand side of (4.4). $\square$

The main theorem of this paper is the following.

Theorem 6. *Let $a_n^{(s,t)} \neq 0$, $b_n^{(s,t)} \neq 0$ be a solution to the discrete 2D Toda molecule (2.1). Let λ be a partition with r rows and c columns, and let $n \geq 0$. Then*

$$\sum_{\pi \in \mathrm{RPP}(\lambda, n)} v(\lambda, n; a, b; \pi) = \prod_{i=1}^{r} \prod_{k=1}^{n} \frac{a_{k-1}^{(r-i, c)}}{a_{k-1}^{(r-i, c-\lambda_i)}}. \tag{4.5}$$

Proof. This theorem is a translation of Theorem 4 via the bijection with the help of (3.6) and Lemma 5. Note that $\mathsf{x}(c) = 0$ for the case of the regular subset $\mathbb{L}(\lambda)$. $\square$

Theorem 6 allows us to find a multiplicative partition function for reverse plane partitions of arbitrary shape with bounded parts from each non-vanishing solution to the discrete 2D Toda molecule (2.1).

5 An example

The discrete 2D Toda molecule (2.1) has the solution

$$a_n^{(s,t)} = [p]_{s+1}^{s+n} (1 - a [p]_1^s [q]_1^{t+n}), \tag{5.1a}$$

$$b_n^{(s,t)} = a [p]_1^{s+n-1} [q]_1^t (1 - [q]_{t+1}^{t+n}) \tag{5.1b}$$

with the notation that $[z]_m^n = \prod_{\ell=m}^{n} z_\ell$ if $m \leq n$, $[z]_m^n = 1$ if $m = n+1$ and $[z]_m^n = \prod_{\ell=n+1}^{m-1} z_\ell^{-1}$ if $m \geq n+2$. The solution involves indeterminates a and p_ℓ, q_ℓ for $\ell \in \mathbb{Z}$ as parameters.

Let λ be a partition with r rows and c columns. Assume that

$$a = [x]_{c-\lambda'_c}^{\lambda_r - r}, \qquad p_i = [x]_{\lambda_{r-i+1}-r+i}^{\lambda_{r-i}-r+i}, \qquad q_j = [x]_{c-j-\lambda'_{c-j}}^{c-j-\lambda'_{c-j+1}} \tag{5.2}$$

The solution (5.1) then turns into

$$a_n^{(s,t)} = [x]_{\lambda_{r-s}-r+s+1}^{\lambda_{r-s-n}-r+s+n} (1 - [x]_{c-t-n-\lambda'_{c-t-n}}^{\lambda_{r-s}-r+s}), \tag{5.3a}$$

$$b_n^{(s,t)} = [x]_{c-t-\lambda'_{c-t}}^{\lambda_{r-s-n+1}-r+s+n-1} (1 - [x]_{c-t-n-\lambda'_{c-t-n}}^{c-t-1-\lambda'_{c-t}}). \tag{5.3b}$$

Let $n \geq 0$. The solution (5.3) yields the weight of (4.3) with (4.2) given by

$$v(\lambda, n; a, b; \pi) = \prod_{\ell=1-r}^{c-1} x_\ell^{\mathrm{tr}_\ell(\pi)} \prod_{(i,j)\in\lambda} \prod_{k=1}^{\pi_{i,j}} \frac{1 - [x]^{j-i-1}_{-n+j+k-1-\lambda'_{-n+j+k-1}}}{1 - [x]^{j-i}_{-n+j+k+\lambda'_{-n+j+k}}}. \tag{5.4}$$

As an instance of Theorem 6 we obtain the following multiplicative partition function for reverse plane partitions.

Theorem 7. *Let λ be a partition with r rows and c columns, and let $n \geq 0$. Then*

$$\sum_{\pi \in \mathrm{RPP}(\lambda,n)} v(\lambda, n; a, b; \pi) = \prod_{(i,j)\in\lambda} \frac{1 - [x]^{\lambda_i - i}_{-n+j-\lambda'_{-n+j}}}{1 - [x]^{\lambda_i - i}_{j-\lambda'_j}} \tag{5.5}$$

where the weight $v(\lambda, n; a, b; \pi)$ is given by (5.4).

Proof. Substituting the solution (5.3) for the right-hand side of (4.5) we get

$$\prod_{i=1}^{r} \prod_{k=1}^{n} \frac{1 - [x]^{\lambda_i - i}_{-k+1-\lambda'_{-k+1}}}{1 - [x]^{\lambda_i - i}_{-k+1+\lambda_i-\lambda'_{-k+1+\lambda_i}}} = \prod_{i=1}^{r} \prod_{j=1}^{\lambda_i} \prod_{k=1}^{n} \frac{1 - [x]^{\lambda_i - i}_{j-k-\lambda'_{j-k}}}{1 - [x]^{\lambda_i - i}_{j-k+1-\lambda'_{j-k+1}}} \tag{5.6a}$$

$$= \prod_{i=1}^{r} \prod_{j=1}^{\lambda_i} \frac{1 - [x]^{\lambda_i - i}_{-n+j-\lambda'_{-n+j}}}{1 - [x]^{\lambda_i - i}_{j-\lambda'_j}}. \tag{5.6b}$$

The last product is the same as the right-hand side of (5.5). $\square$

The multiplicative partition function in Theorem 7 generalizes the multi-trace generating function (1.2) by Gansner. Indeed (5.5) reduces into (1.2) as $n \to \infty$ because $\lim_{n\to\infty} [x]^{\mathrm{const.}}_{-n+\mathrm{const.}} = 1$ as formal power series, $\lim_{n\to\infty} \lambda'_{-n} = r$ and $\lim_{n\to\infty} v(\lambda, n; a, b; \pi)$ $= \prod_{\ell=1-r}^{c-1} x_\ell^{\mathrm{tr}_\ell(\pi)}$.

Assuming $x_\ell = q$ for all $\ell \in \mathbb{Z}$ we obtain the partition function

$$\sum_{\pi \in \mathrm{RPP}(\lambda,n)} v(\lambda, n; a, b; \pi) = \prod_{(i,j)\in\lambda} \frac{1 - q^{\lambda_i + \lambda'_{j-n} - i - j + n + 1}}{1 - q^{\lambda_i + \lambda'_j - i - j + 1}} \qquad \text{with} \tag{5.7a}$$

$$v(\lambda, n; a, b; \pi) = q^{|\pi|} \prod_{(i,j)\in\lambda} \prod_{k=1}^{\pi_{i,j}} \frac{1 - q^{n-i-k+1+\lambda'_{-n+j+k-1}}}{1 - q^{n-i-k+1+\lambda'_{-n+j+k}}} \tag{5.7b}$$

from (5.4) and (5.5). If $\lambda = (c^r)$, an $r \times c$ rectangular shape, that becomes the product formula (1.1) by MacMahon. The partition function (5.7) is thus regarded as a generalization of (1.1) for reverse plane partitions of arbitrary shape.

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